

# EXPERT MATHEMATICIANS' STRATEGIES FOR COMPARING THE NUMERICAL VALUES OF FRACTIONS – EVIDENCE FROM EYE MOVEMENTS

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*There has been a controversial debate if individuals solve fraction comparison tasks componentially by comparing the numerators and denominators, or holistically by considering the numerical magnitudes of both fractions. Recent research suggested that expert mathematicians predominantly use componential strategies for fraction pairs with common components and holistic strategies for pairs without common components. This study for the first time used eye movements to test if this method allows distinguishing strategy use on specific problem types in expert mathematicians. We found the expected fixation differences between numerators and denominators in problems with common components but not in problems without common components.*

## THEORETICAL FRAMEWORK

Numerous studies have shown that students at all age levels experience large difficulties with learning of and dealing with fractions (e.g., Vamvakoussi & Vosniadou, 2004). A typical mistake is to consider a fraction as two separate natural numbers (the numerator and the denominator) rather than as one rational number. Accordingly, students have been found to compare two fractions by comparing their components separately rather than by comparing the holistic fraction values. As a consequence, many students make typical mistakes when componential comparison is not in line with holistic comparison (Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). For example, they believe that  $\frac{1}{4}$  is larger than  $\frac{1}{3}$  because 4 is larger than 3.

Further studies have suggested that adults also tend to base their comparison of fractions on the fractions' natural number components ("natural number bias"; Vamvakoussi, Van Dooren, & Verschaffel, 2012), and that even expert mathematicians do so in special cases of comparison problems, namely when the two fractions have the same numerator or the same denominator (Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013). In fact, in such cases, it can be an effective strategy to compare only the non-equal components of the fractions, rather than taking into account the fraction magnitudes.

These experiments with fraction comparison problems also contributed to the debate on whether individuals generally process fractions componentially by focusing on their components, or holistically by taking into account the fraction values. While Bonato, Fabbri, Umiltà, and Zorzi, (2007) suggested componential processing, Schneider and Siegler (2010) showed that holistic processing is also possible, and Meert, Grégoire,

and Noël (2010) proposed a hybrid model of fraction processing including componential and holistic characteristics. As suggested by Obersteiner et al. (2013), the question whether a person applies a componential or holistic strategy might crucially depend on individual factors and specific task characteristics. In a computerized experiment involving expert mathematicians, these authors could show that response times on fraction comparison items depended on the numerical distance between the numerators when the denominators were equal (e.g.,  $16/21$  vs.  $20/21$ ), and on the numerical distance between the denominators when the numerators were equal (e.g.,  $4/17$  vs.  $4/39$ ). When the two fractions had no common components (e.g.,  $11/23$  vs.  $19/31$ ), response times depended on the difference between the fraction values rather than on the differences between the components. These results led to the conclusion that expert mathematicians take into account the fraction values only when no easier strategy (comparing the components) is applicable.

The conclusions above were based on response time data that were recorded in a computerized experiment and averaged across participants. Although such a method has certain advantages, it is only an indirect measure of individual strategies. The reason is that it is not possible to control for all factors that might have influenced response times in addition to the numerical distances between fractions or fraction components, so that alternative explanations (e.g., specific task features) for the observed response time patterns cannot be completely ruled out. Also, response time data on the group level do not take into account that strategy use might vary largely between individuals. In fact, there is a large number of strategies that can be applied to fraction comparison problems, and there is empirical evidence that individuals indeed make use of a wide range of strategies (Clarke & Roche, 2009).

### **Eye Movements as a Method for Assessing Individual Strategies**

Assessing individual strategies on cognitive problems is a methodological challenge. Individual reports have been used in previous research, but the reliability of this method can be questioned, in particular in younger participants (Robinson, 2001). Recently, recording eye movements has become more and more attractive to researchers to assess individual strategies on mathematical tasks. For example, Green, Lemaire, and Dufau (2007) could show that eye movements were a reliable measure of individual strategies in multi-digit addition problems; Sullivan, Juhasz, Slattery, and Barth (2011) successfully used eye movements to assess adults' strategies on positioning numbers on a number line; and Dewolf, Van Dooren, Hermens, and Verschaffel (2013) used eye movements to validate students' strategies on mathematical word problems. In these and many other studies, recording eye movement has been considered a promising tool for investigating individual strategies, because eye fixations and eye movements are assumed to correspond to mental operations (Grant & Spivey, 2003).

## **THE PRESENT STUDY – QUESTIONS AND HYPOTHESES**

The aim of the present study was to test if recording eye movements could be a suitable method for assessing individual strategies on fraction comparison problems. Following the results of the above-mentioned study by Obersteiner et al. (2013), we addressed the questions if expert mathematicians indeed solve fraction comparison problems by comparing the numerators when the denominators are equal; by comparing the denominators when the numerators are equal; and by comparing the fraction magnitudes when the fractions do not have common components; and if these strategies could be measured through eye movements. We involved expert mathematicians in this study, because for these people it was possible to establish clear hypothesis concerning the strategies they would use for comparing fractions, based on previous studies. This would not have been the case for students who have been found to apply a variety of strategies, many of which are actually invalid (Clarke & Roche, 2009). As this was – to the best of our knowledge – the first time eye movement was used for assessing strategies in fraction comparison, the aim of our study was to show that this method was in principle suitable for this purpose.

We recorded individual fixation times of both eyes and hypothesized that fixation times would be longer for numerators than for denominators when the fractions have common denominators (Hypothesis 1), because participants would have to spend more time on comparing the numerical values of the (non-equal) numerators than to verify that the denominators are equal. For the analogue reason, we hypothesized that fixation times would be longer for denominators than for numerators when the fractions have common numerators (Hypothesis 2). Finally, we expected that fixation times would be equally long for denominators and numerators when the fractions do not have common components (Hypothesis 3), because the participants would need to take into account the numerical values of all numbers involved to determine the fraction magnitudes, and it is not sufficient to compare the components separately.

## **METHOD**

### **Participants**

There were eight participants in this study with high expertise in mathematics. Six of them were staff members of a German university who had an academic degree in mathematics, and two were students majoring in mathematics. The mean age of these eight participants (five female) was 26 years ( $SD = 3.9$ ).

### **Design and Procedure**

The participants sat in front of a computer screen, which was connected to a binocular remote contact free eye tracking device (SensoMotoric Instruments) with a sampling rate of 500 Hz. The eye tracking device was placed underneath the screen. The participants were asked to avoid head and body movements as far as possible. First, calibration was performed through fixations of nine small dots on the screen. After that, two practice trials were presented to make the participants familiar with the

procedure. Then the experiment started, and two fractions at a time appeared next to each other. Participants were instructed to choose the larger fraction as fast and accurately as possible by saying aloud the word “left” or “right”. Their answers were noted down by a researcher who supervised the experiment. After each trial, a fixation cross appeared in the middle of the screen for two seconds.

All in all, there were 32 fraction comparison items, half of which had common components (eight pairs with common numerators, eight pairs with common denominators). To be consistent with the experiment conducted by Obersteiner et al. (2013), we presented the items with common components and the items without common components in two separate blocks. Within each block, items appeared in pseudo-randomized order.

## RESULTS

Data from one participant had to be excluded from the analysis due to low calibration quality. To analyse fixation times on numerators and denominators, we defined rectangular-shaped same-sized areas of interest (AOI) that surrounded the numerators (AOI “Num”) or the denominators (AOI “Denom”) of both fractions (Figure 1). We then compared fixation times between these AOIs for each fraction type.

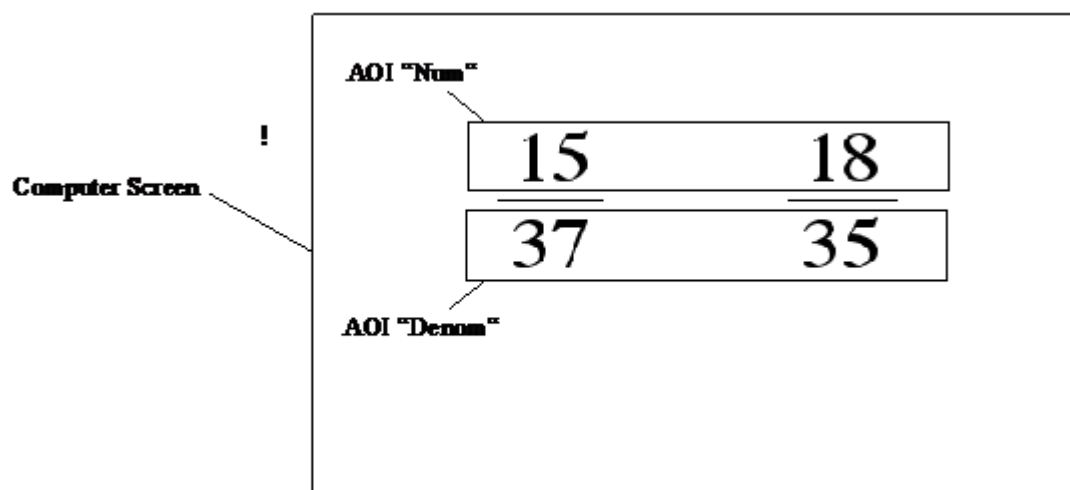


Figure 1: Sample item and areas of interest (AOI).

Table 1 displays the mean fixation times in ms for numerators and denominators for each fraction type. For the statistical comparison of fixation times between AOIs, we used a generalized estimating equation model that takes into account repeated measures within subjects.

Type	Numerators		Denominators	
	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>
Common Denominators	359	264	185	189
Common Numerators	391	459	561	504
Without Common Components	1053	836	820	586

Table 1: Mean fixation times (in ms) for numerators and denominators, depending on comparison type. Note: *M* = Mean, *SD* = Standard deviation.

As expected, for items with common denominators, fixation times were significantly higher for numerators than denominators, Wald  $\chi^2(1, N = 7) = 21.47, p < .001$ , suggesting that participants paid more attention to the (unequal) numerators than to the (equal) denominators, which supports Hypothesis 1. For items with common numerators, fixation times were significantly higher for denominators than for numerators, Wald  $\chi^2(1, N = 7) = 5.76, p = .016$ , suggesting that participants focused more on the (unequal) denominators than on the (equal) numerators. This is in line with Hypothesis 2. For the items without common components, the difference between fixation times for numerators and denominators was not significant, Wald  $\chi^2(1, N = 7) = 2.28, p = .131$ , supporting Hypothesis 3.

These results are in line with our expectation that the participants in our study would apply componential comparison strategies to fraction comparison problems with common components, and holistic comparison strategies to comparison problems without common components. Indeed, fixation patterns as illustrated by heat maps (Figure 2) lend further support to this assumption. Figure 2 displays heat maps for three selected items. In heat maps, reddish colours indicate longer fixation times. For the items with common denominators (2a) or common numerators (2b), fixations were predominantly placed on the non-equal parts of the fractions. For the item without common components, (2c), the heat maps indicate that fixations were more equally distributed among the fractions' components, and they suggest that participants spent more time on comparing each fraction's numerator and denominator, hinting to a holistic approach, in which the numerical value of each fraction is determined through the numerical relation between numerator and denominator.



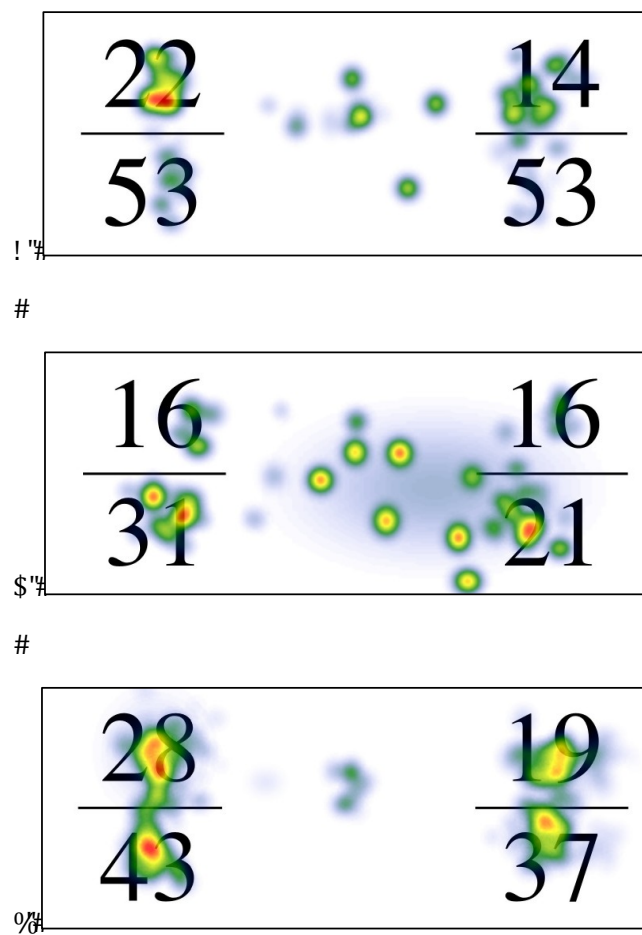


Figure 2: Heat maps for sample items with common denominators (a.), common numerators (b.) and without common components (c.) *Note:* Reddish colours indicate longer fixation times.

## DISCUSSION

This study was the first to report eye movement data during fraction comparison. We involved adults with high expertise in mathematics so that we could establish clear hypotheses concerning their strategies on specific types of fraction comparison problems, as reported in the literature. We distinguished comparison problems with fraction pairs that have common numerators, common denominators, or no common components. The purpose of this study was to investigate whether the expected differences in strategy use between problem types could be assessed through eye movements.

In line with the results of the computerized experiment conducted by Obersteiner et al. (2013), the data suggest that the participants in our study focused on the non-equal components of the fractions when the two fractions had common components, but that they used a holistic approach when the two fractions did not have common components. This result helps understanding the controversial conclusions that have been drawn from studies about individuals' strategies in fraction comparison (e.g., Bonato et al., 2007; Meert et al., 2010; Obersteiner et al., 2013; Schneider & Siegler,

2010; Vamvakoussi et al., 2012; Van Hoof, Lijnen, et al., 2013; Van Hoof, Vandewalle, & Van Dooren, 2013). It supports the assumption that adults with high expertise in mathematics use different strategies for comparing the numerical values of two fractions, and that these strategies depend on the specific type of fraction comparison task at hand. When the fractions have common components, they prefer componential strategies; when the fractions do not have common components, they prefer holistic strategies. As the participants in our study were expert mathematicians, this conclusion might not generalize to other individuals such as primary and lower secondary school students. However, when studying performance on fraction comparison problems, one should always be aware that individuals could apply different strategies depending on the type of item.

Concerning the method of our study, we can conclude that recording eye movements is a promising tool to assess individual strategies. It might be used especially fruitfully with participants and on tasks for which self-reports are less reliable. More specifically, recording eye movements on fraction comparison could allow assessing the large variety of strategies that students have been reported to use on such problems (Clarke & Roche, 2009). The present study can be seen as a first step towards further investigations of eye movements on fraction problems in adults without mathematical expertise and – more importantly – in school students. Assessing these strategies can also be useful for identifying typical misconceptions about fractions that students might have. This could eventually lead to teaching approaches that are tailored to the individual needs of students.

A limitation of the present study is certainly the low sample size, which limits the generalizability of our findings. We are currently conducting a follow-up study with very similar items in a larger sample of students of mathematics to replicate the results presented here. Moreover, further analyses on the individual level could allow deeper insight into individuals' fraction comparison strategies.

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